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Karmelita H. Pjanić[1]
University of Bihać, Pedagogical faculty, Bosnia and Herzegovina

Short scientific paper

Use of Abu'l-Wafa's problem to foster pedagogical knowledge of prospective mathematics teachers

Summary: *Solving Abu'l-Wafa's problem could be a powerful tool for building and fostering pedagogical content knowledge of prospective mathematics teachers. The goal of this case study is to examine if this episode in the history of mathematics would foster the subject content knowledge and pedagogical content knowledge of a group of prospective mathematics teachers.*

Abu'l-Wafa's problem was presented to five prospective mathematics teachers in the Mathematics Teaching Methodology Course. They had to find out at what point of learning geometry and how to engage pupils to solve this problem, taking into account the question of procedural and conceptual knowledge in mathematics as well as the important question of the role of proof and argumentation in mathematics classes. Our case study showed that an integration of the history of mathematics in education may be particularly relevant for supporting and improving the pedagogical content knowledge of prospective mathematics teachers.

Keywords: *Abu'l-Wafa's problem, mathematical and pedagogical content knowledge, prospective mathematics teachers.*

Introduction

In the process of education, learners need to learn about the past and present in order to be able to perceive the flows of the future. The optimal ratio of the past and present in the educational process should be determined (Poljak, 1991). Many years of history of mathematics are behind us. Yet, just because history of mathematics is so extensive, it is important to be able to single out what is essential to the content of learning mathematics. Based on the author's experience, mathematics teachers are aware of the principles of historicity, but they generally understand and apply it in the sense of telling short episodes about the path of developing a math-

1 kpjanic@gmail.com

ematical idea or an anecdote from the life of a mathematician. The question arises as to why mathematics teachers limit the use of mathematics history in teaching using only these two aspects.

Episode from history of mathematics in mathematics classes

Research (Ball, 1988; Cooney at al, 1998; Furinghetti, 2007; Philippou & Christou, 1998) suggests that teachers teach in the manner in which they were taught. Thus, the teachers' presentation and discussion of mathematical concepts are largely affected by their past experience as students. This implies that the teachers who have had relatively little or no exposure to history of mathematics during their secondary and higher education, may perceive themselves lacking expertise and consequently not expose their students to the history of mathematics. Mathematics history is an excellent source of interesting problems that supply opportunities to sharpen problem-solving skills (Swetz, 1986). History offers diverse approaches, as well as a variety of algorithms and techniques for solving similar problems that allow students to develop skills such as considering multiple strategies and evaluating both solutions and procedures. Using historical problems in instruction can bring up related problems, thereby stimulating further mathematical explorations and discussions. History is full of mathematical connections: connections between mathematical topics, connections between mathematics and applications, connections between mathematics and other disciplines. In addition, history is full of mathematical connections across centuries, cultures, and regions of the globe.

As Pjanić (2019) proposed, solving Abu'l-Wafa's problem could be a powerful tool for building and fostering the subject content knowledge and pedagogical content knowledge of pre-service mathematics teachers. Abu'l-Wafa in his treatise On Those Parts of Geometry Needed by Craftsmen described several constructions made with the aid of ruler "and rusty compass", a compass with a fixed angle. These included constructing a perpendicular at the endpoint of a line segment, dividing the segments in equal parts, bisecting angles, constructing a square in a circle, and constructing a regular pentagon (Berggren, 2003).

> **Problem:** *Construct at the endpoint A of segment AB a perpendicular to that segment, without prolonging the segment beyond A.*
>
> As Pjanić (2019) demonstrated, the problem may be solved in different ways. This problem could be posed to pupils of different age either in middle or high school.
>
> We will show the effects of presenting this problem on the subject and pedagogical content knowledge of a group of prospective mathematics teachers.

Methodology

In this paper, we refer to a course of methodology of teaching mathematics for prospective mathematics teachers in which the episodes from the history of mathematics were introduced as mediators of knowledge for teaching. Mathematics Teaching Methodology Course is a compulsory course in the fourth year of the undergraduate teacher education program. The general idea was to offer a role model for integrating history of mathematics in mathematics classes.

The goal of this study is to examine if the use of an episode from the history of mathematics fosters the subject content knowledge and pedagogical content knowledge of a group of prospective mathematics teachers. Specifically, we want to determine whether the student will perceive different connections between concepts, whether they will provide multiple solutions and proofs, how they will create didactic situations, including the given problem.

Our study is set in a prospective undergraduate teacher education program at the University of Bihać aimed at specializing in teaching mathematics and physics or both. The full program lasts four academic years; it focuses on mathematics and physics courses, followed in a much smaller amount of teaching hours by psychology, didactics, mathematics education, and physics education. In the second year there are courses on the foundations of the history of natural sciences. However, the history of mathematics is an elective course in the master program. In the undergraduate program, the prospective teachers spend some weeks in practical training at elementary and high schools and they are assisted by experienced teachers: at the beginning simply as observers and afterwards also acting as teachers (they assign tasks, deliver lectures, assess students' performance). All course participants have no teaching experience.

This integration of history in the methodology of teaching mathematics does not require that the participants' teaching sequences and didactic situations should include historical segments; rather it requires that history should inspire teaching strategies. The phases of the experiment were:

1. Discussion about learning trajectory that includes notions of bisector, perpendicular, angle and circle.
2. Presentation of the episode from the history of mathematics: Abu'l Wafa's problem.
3. Homework: Students had to analyse the mathematical programs for elementary and high school and to look at some textbooks commonly used in elementary and high schools.
4. Homework: Students had to solve the problem and design the teaching sequence/ didactic situation that include the problem. The sequences designed by the prospective teachers had to fulfill the program requirements and be suitable for the relevant school level.
5. Discussion among students and confrontation of the produced learning sequences / didactic situations. In this phase the focus is on linking different mathematics ideas and stressing the possibilities of a rich fundus of mathematics history that can be used in the process of teaching mathematics.

Results and discussion

During the first phase of the experiment students had to check the mathematics program for elementary and high schools and the related topics in the textbooks and they agreed that teaching about perpendicular, bisector, angles and circle in the middle school is focused on developing pupils' procedural thinking. After the Abu'l Wafa's problem had been introduced to five prospective mathematics teachers during the sessions of the Mathematics Teaching Methodology Course, they had to do several tasks for homework. At first, they analyzed the mathematics program for elementary and high schools, as well as the textbooks commonly used in math classes in order to find connections with the given problem. They found the solutions to the problem and proposed similar problems. Furthermore, they proposed didactical situations and teaching sequences based on the given problem. The students had three working days to submit the homework. We will present the students' solutions and proposals that were discussed in the fifth phase of the experiment.

The analyses of the textbooks for elementary and high school[2] showed that neither the Abu'l Wafa's problem nor the similar problems are included in the textbooks.

All five students solved the problem, i.e., constructed the perpendicular at the endpoint of the

2 The list of this textbooks is not a problem of this article, so we will omit it.

segment line and proved the construction, in three ways:

1) Construction of right angle at vertex A (Figure 1);

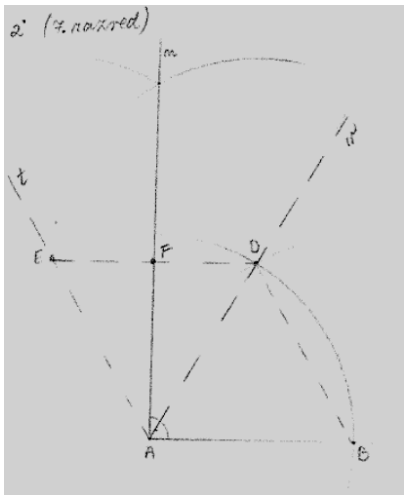

Figure 1. Construction of right angle at vertex A.

2) Construction of bisector of AB and translation of that bisector to the endpoint A (Figure 2);

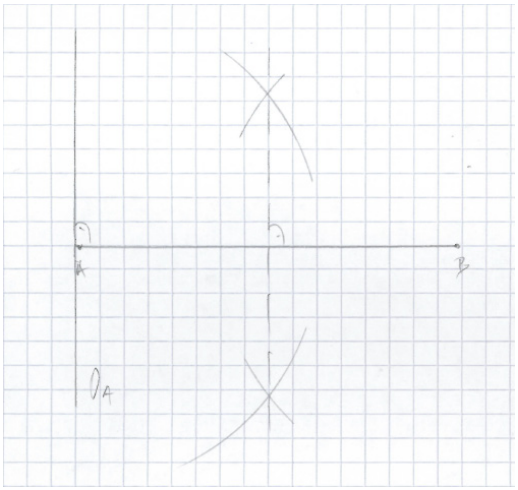

Figure 2. Construction of bisector of AB and translation to the endpoint A.

3) Abu'l Wafa's solution with proof in terms of central and peripheral angle of circle (Figure 3). Two students additionally proved Abu'l Wafa's construction using the characteristics of the triangle.

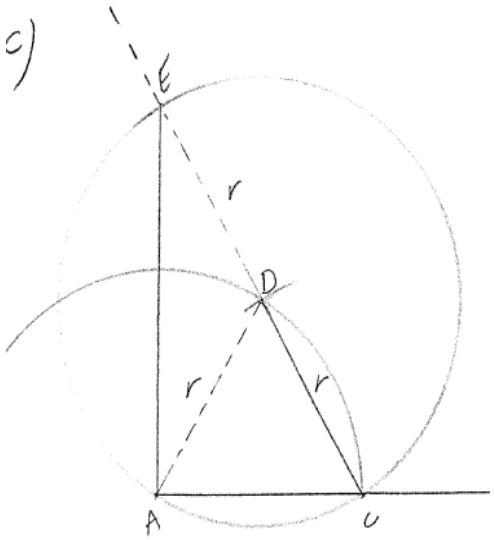

Figure 3. Abu'l Wafa's solution.

One of the students suggested the fourth solution (Figure 4). This student solved the problem by using the characteristics of the regular hexagon.

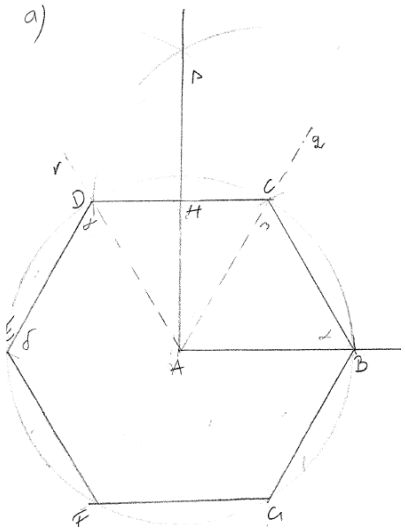

Figure 4. Construction of the regular hexagon.

Based on their solutions, the students proposed that Abu'l Wafa's problem could be posed to the middle grade pupils when they learn about the construction of the perpendicular, bisector and right angle, and also when they learn translation. They proposed that the next problem could be posed to middle school pupils as an auxiliary problem: *Construct the perpendicular at the endpoint of given segment line.* The condition of not extending the segment line beyond the end point is omitted here. Figure 5 illustrates one of the solutions of the auxiliary problem.

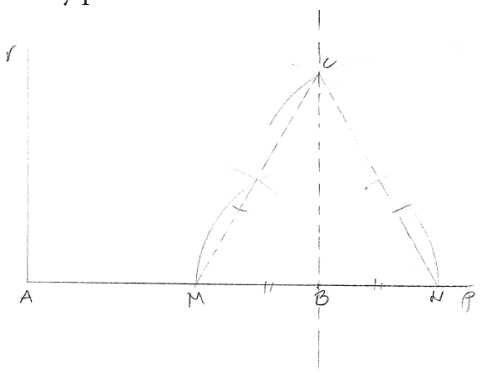

Figure 5. Solution of the auxiliary problem.

As the goals of solving this task in middle school, the students emphasized practicing the construction of perpendicular, right angle or translation (developing procedural knowledge), but also reinforcing conceptual knowledge about those geometric notions. On the other hand, all five students suggested that Abu'l Wafa's problem gives an opportunity to high school pupils to practice argumentation and proof. However, they were not sure how to incorporate Abu'l Wafa's solution in the teaching process at the high school level. During the discussion phase, the students agreed that a learning situation in high school could be created in such a manner that teacher present the Abu'l Wafa's problem and ask pupils to solve it. The students assumed that the high school pupils could not link the solution of the problem to the central and peripheral angles by themselves. Accordingly, they suggested that teachers should present Abu'l Wafa's construction and ask pupils to prove it. The discussion ended with the final remark that the teacher should not rely solely on the textbook when creating the learning sequence and didactic situations. Textbook content should be the starting point to teachers in finding the possibilities of creating learning situations, and the history of mathematics offers such possibilities.

Conclusion

We argue that integrating episodes of history of mathematics into the mathematics classes could be beneficial to pupils as well as to students of mathematics education. Rather than become an additional task, it can be a tool for effective teaching both in programs of methodology of teaching mathematics for future teachers as well as in school mathematics. In our study, prospective mathematics teachers presented five solutions of the given problem from the history of mathematics. Furthermore, they proposed didactical situations that include Abu'l Wafa's problem.

As previous research suggested, the teachers' presentation and discussion of mathematical concepts are largely affected by their past experience as students (Ball, 1988; Cooney et al, 1998; Furinghetti, 2007; Philippou & Christou, 1998). According to this, by introducing episodes from the history of mathematics into the course of Methodology of teaching mathematics, we have exposed a group of students to the history of mathematics as the source of problems upon which we can create a teaching process. Our case study showed that integration of the history of mathematics may be particularly relevant to support and improve didactical background of prospective mathematics teachers. Namely, students who participated in the study succeeded to perceive different connections between concepts, provided multiple solutions and proofs and created didactic situations, including the given problem.

Mathematics teachers need to find a "right measure" in teaching paying attention both to performing mathematical procedures as well as to give explanations, argumentations and proofs. They have to promote such mathematical culture among their pupils. Using episodes from history of mathematics could help teachers to accomplish this (Pjanić, 2019).

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Кармелита Х. Пјанић

Педагошки факултет, Универзитет у Бихаћу, Босна и Херцеговина

УПОТРЕБА ПРОБЛЕМА АБУЛ-ВЕФЕ ЗА ПОДСТИЦАЊЕ МЕТОДИЧКОГ ЗНАЊА БУДУЋИХ НАСТАВНИКА МАТЕМАТИКЕ

Наставници математике морају да савладају математичке садржаје, као и да стекну педагошка знања која ће им омогућити да стварају окружење у којем ће ученици учити на најбољи и најефикаснији начин. У настави математике важно је да код ученика буду подједнако развијена концептуална и процедурална знања. Наставници математике треба да нађу „праву меру" у подучавању и да посвете пажњу како извођењу математичких поступака, тако и давању објашњења, аргумената и доказа (Pjanić, 2019). Управо такву математичку културу наставници треба да развијају међу својим ученицима. У томе им може помоћи коришћење појединих епизода из историје математике.

Решавање проблема Абул-Вефе могло би да буде ефикасно средство за изградњу и неговање методичког знања будућих наставника математике. У свом трактату под називом О деловима геометрије који су потребни занатлијама (On Those Parts of Geometry Needed by Craftsmen) Абул-Вефа описује неколико конструкција направљених уз помоћ лењира и „зарђалог компаса", тј. компаса са фиксираним углом. У ове конструкције спада конструисање нормале на крајњој тачки праве, подела сегмената на једнаке делове, одређивање симетрале угла, цртање квадрата унутар круга и конструисање правилног петоугла (Berggren, 2003).

Циљ ове студије случаја је да испита да ли би ова епизода из историје математике подстакла математичка и методичка знања групе будућих наставника математике. Проблем Абул-Вефе представљен је групи од петоро будућих наставника математике током предавања из предмета Методика наставе математике. Експеримент се састојао од неколико фаза:

- Дискусија о плану учења који обухвата стицање знања о појмовима симетрале, нормале, угла и круга;

- Презентација на тему једне епизоде из историје математике: Проблем Абул-Вефе;

- Домаћи задатак: Студенти треба да анализирају наставне програме математике за основну и средњу школу и да прегледају неке од уџбеника који се обично користе на овим нивоима;

- Домаћи задатак: Студенти треба да реше овај проблем и да представе ток часа/дидактичку ситуацију у коју ће бити укључен овај проблем. Ток часа морао је да буде у складу са захтевима наставног програма и примерен одређеном школском нивоу;

- Дискусија међу студентима и упоређивање дидактичких ситуација/тока часа које су припремили. У овој фази фокус је на повезивању различитих математичких идеја и на наглашавању могућности за употребу богатог фундуса историје математике у настави математике.

Ова група будућих наставника требало је да открије у ком стадијуму учења геометрије и на који начин је потребно укључити ученике у решавање овог проблема, узимајући у обзир питање процедуралног и концептуалног знања из математике, као и важно питање улоге доказа и аргументације у настави математике.

Учесници из студије случаја понудили су пет различитих решења за задати проблем: конструисање правог угла при темену A, конструисање симетрале AB и транслација ка крајњој тачки A, Абул-Вефино решење – веза централног и периферног угла, конструкција правилног шестоугла, предлог и решење помоћног проблема који је довео до решења Абу-Вафиног проблема.

У току фазе која је подразумевала дискусију, студенти су се сложили да би ситуација за учење на средњошколском нивоу могла да буде креирана тако да наставник представи ученицима проблем Абул-Вефе, а потом тражи од ученика да га реше. Студенти су претпоставили да ученици средње школе не могу да повежу решење проблема са централним и периферним угловима. Сходно томе, предложили су да наставници прво представе ученицима конструкцију Абул-Вефе, а онда да траже од њих да је докажу.

Наша студија случаја показала је да интеракција историје математике у (методичком) обучавању може да буде посебно релевантна за подршку и побољшање методичких знања будућих наставника математике.

Будући наставници математике који су учествовали у овој студији случаја успешно су уочили различите везе између појмова, представили су неколико решења и доказа, дискутовали су о различитим аспектима проблема и довели их у везу са могућим решењима. На крају, креирали су дидактичке ситуације у које је укључен математички проблем Абул-Вефе.

Кључне речи: *проблем Абул-Вефе, математичко и методичко знање, будући наставници математике.*