Teaching Innovations, Volume 33, Issue 1, pp. 36–56
doi: 10.5937/inovacije2001036L

Charalampos E. Lemonidis[1]
University of Western Macedonia, School of Social Sciences and Humanities, Department of Educational Elementary Studies, Florina, Greece

Anastasios C. Gkolfos
4th High School of Evosmos, Thessaloniki, Greece

Scientific review paper

Paper received: Nov 13 2019
Paper accepted: Jan 20 2020
Article Published: Mar 30 2020

Number line in the history and the education of mathematics

Extended summary

In this study we first attempt to present relevant research and analyze the types of number lines, then present two studies examining components of number line, namely the direction and negative numbers based on historical evolution. Then a historical analysis of the evolution of number line is presented. After that, we present the difficulties encountered by students in using the number line as a representation tool, and through the historical evolution of the number line we try to contrast these difficulties with critical points in its historical evolution.

1st period. Mathematics up to Euclid: Separation of numbers and lines.

In Greek mathematics there was a clear separation between number and magnitude. Numbers (natural numbers) were simply collections of discrete units that measured a multitude. Magnitude Size, on the other hand, was usually described as a continuous quantity divided into parts and is infinitely divisible. This distinction between number and magnitude also resulted in the distinction between arithmetic and geometry. Arithmetic dealt with a discrete or non-extended quantity, while geometry dealt with a continuous or extended quantity. It also imposed a different way of solving and handling many problems.

1 xlemon@uowm.gr

2nd period. Until the 16th Century: Foundations of Integers - Rational Numbers - Empirical Geometry

From the 12th to the 16th century an orientation in empirical Geometry is observed, as well as its relation to computational methods and the use of measuring tools. Michael Stifel (1487-1567) was the first to define negative numbers as numbers less than zero and positive numbers as greater than zero. He was the first to describe zero, fractional and irrational numbers as numbers. In the 16th century a transformation began in the classical conception of number and magnitude. Francois Viete (1540-1603) introduced a new form of symbolism to denote unknown magnitudes and numbers, stating that numbers and magnitudes can be interchanged. This relationship between numbers and magnitudes encouraged the idea that numbers could also be treated as if they were continuous, in the Aristotelian sense of continuity. However, even during this period there is no concept of the number line and the identification of each point with a number. Nevertheless, the elements that make it possible to display it are beginning to emerge.

3rd period. From the 17th century to the beginning of the 19th century: The first connection of numbers and geometric lines - Algebraization of geometry

Mapping lines of numbers was not a common idea among mathematicians until the end of the 16th century. However, the concept of the number line begins to emerge in the 17th century by some pioneering mathematicians. John Wallis (1685) was the first to use a number line in his book, *Treatise of Algebra*, in order to interpret addition and subtraction with negative numbers. The development of algebraic symbolism and the connection of curves to their equations led Descartes to the algebraization of geometry with the help of the coordinate system. Descartes does not use the terms abscissa, ordinate or axis in his work. Descartes did not introduce the number line through the discovery of the coordinate system in 1637 in his work La Géometrie, as he never mentions the concept of axis, and none of his illustrations depicts an axis or a numerical system of coordinates, coordinates, even when values for specific magnitudes are specified. The first recording of number lines appears in the first half of the nineteenth century in the work of Ernst Gottfried Fischer (1754-1831). Fischer works with negative and positive quantities without limitation. He explicitly associates each point on the axis of the abscissa with the values of x and corresponds to one point on the axis of the ordinate.

4th period. From the beginning of the 19th century to the present: The foundation of the number line in its present form.

The first attempt to develop a theory of real numbers was made in the early 1830s by Bolzano, who saw real numbers as limits of progressions of rational numbers. Around the same time, Rowan Hamilton (1805-1865) made an attempt to define real numbers, but could not escape the logic of ordinary geometric tradition. Karl Weierstrass (1855-1897), wanting to base infinite calculus solely on the concept of number, believed that he had to define irrational numbers irrespective of the concept of limit. He therefore considered the convergent sequence itself as the number or the limit. Thus, irrational numbers are defined as sets of rational numbers, rather than ordered sequences of rational numbers. In 1871 Georg Cantor (1845-1918), launches a new numbering program, similar to the Meray and Weierstrass programs. At the same time Heine (1821-1881) proposed some simplifications that led to the so-called Cantor-Heine development, which resembles that of Meray, in which convergent sequences that do not converge to rational numbers are considered to define irrational numbers.

Dedekind tried to give a clear definition of continuity, first for the points of a straight line and then for a set of numbers starting from the set of rational numbers, after observing that the ordering properties of rational numbers apply just as the relations between the points of a straight line. Dedekind considered that the set of rational numbers could be extended to a continuous set of real numbers if the Cantor - Dedekind principle accepted that the points of a straight line could be mapped one by one with the real numbers. Therefore, we have the foundation of the number line in its current form.

First of all, we should note that the mathematical integration and constitution of the notion of the number line, as we know it today, took place very slowly in the history of mathematics. As we have seen, it was only with the foundations of Dedekind and Cantor in the late 19th and the early 20th century that we can now consider that there is a one-to-one correspondence between the points of a straight line and the numbers of the set of real numbers. This in itself shows that while the concept of the number line appears to be simple, its composition to its present form has taken a long time.

As critical points in this mathematical constitution of the notion of the number line over the course of mathematical history, we can in principle consider the separation between the numbers and the magnitude or the separation between the numbers and the straight line. We can see this separation, as we indicated above, in the mistakes of students who often manage numbers separately from the measures on the straight lines of numbers.

The second critical point that appears in the historical evolution, but also as a difficulty for students, is the negative numbers and the orientation on the number line in the positive or negative direction.

The third critical point is the density of rational numbers and the extra unit intervals needed to place them on the line of numbers. This manifests as a problem for students, when they have to place fractions, and generally rational numbers, on the number line where it is necessary to determine the extra unit space such as, for example, specifying the unit of ¼ to place ¾ on the number line.

The fourth critical point focuses on the density of the irrational numbers, the separation of the rational from the irrational numbers and the representation of the irrational numbers on the number line.

In this work we examined the difficulties of students and teachers of mathematics when dealing with the density and separation of rational and irrational numbers and the representation of irrational numbers on the number line. As it was pointed out earlier and drawing on historical information, the identification of real numbers and their correspondence with points on the number line reaches its conclusion very late, actually between the late 19^{th} and the early 20^{th} centuries.

Keywords: Number line, historical evolution, epistemological obstacle, concept representation.

References

- Amadeo, M. (2018). Textbooks revealing the development of a concept – the case of the number line in the analytic geometry (1708–1829). *ZDM*, 50 (5), 907–920. DOI: 10.1007/s11858-018-0968-7
- Bachelard, G. (1938). *La formation de l'esprit scientifique*. Paris: J Vrin.
- Beishuizen, M. (1999). The empty number line as a new model. In: Thompson, I. (ed.) *Issues in teaching numeracy in primary schools* (157–168). Buckingham: Open University Press.
- Boyer, C. B. & Merzbach, U. C. (1997). *Η ιστορία των μαθηματικών.* Αθήνα: Πνευματικός. DOI: 10.2307/2532593 [The history of mathematics] [in Greek].
- Bright, G. W., Behr, M. J., Post, T. R. & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics Education*, 215–232. DOI: 10.2307/749066
- Brousseau, G. (1976). Les obstacles épistémologiques et les problèmes en mathématiques. In: *Comptes-rendus de la XXVIIIème rencontre organisée par la Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (101–117). Louvain-la-Neuve.
- Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Rech Didact Math*, 4 (2), 165–198.
- Bunt, L. N. H., Jones, P. S. & Bedient, J. D. (1981). *Οι ιστορικές ρίζες των στοιχειωδών μαθηματικών.* Αθήνα: Πνευματικός. [The historical roots of elementary mathematics] [in Greek]
- Chambers, E. (1728). *Cyclopaedia; or An universal dictionary of arts and sciences* (Vol. 1–2). London: Printed for James and John Knapton.
- Clarke, D. M., Roche, A. & Mitchell, A. (2007). Year six fraction understanding: A part of the whole story. In: Watson, J. & Beswick, K. (Eds.). *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia (MERGA): Mathematics: Essential research, essential practice Vol. 1* (207–216). Sydney: MERGA.
- Crossley, J. N. (1987). *The Emergence of Number*. World Scientific Publishing Co Pte Ltd. DOI: 10.1142/0462
- Dickinson, P. & Eade. F. (2004). Using the number line to investigate the solving of linear equations. *For the Learning of Mathematics*, 24 (2), 41–47.
- Diezmann, C. M., Lowrie, T., & Sugars, L. (2010). Primary students' success on the structured number line. *Australian Primary Mathematics Classroom*, 15 (4), 24–28.
- Diezmann, C. M. & Lowrie, T. (2006). Primary students' knowledge of and errors on number lines. In: Grootenboer, P., Zevenbergen, R. & Chinnappan, M. (Eds.). *Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia: Identities, cultures, and learning spaces* (171–178). Sydney: MERGA.
- Duroux, A. (1983). La valeur absolue: difficultés majeures pour une notion mineure. *Petit x*, 3.
- Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In: Hitt, F. & Santos, M. (Eds.). *Proceedings of the Twenty First Annual Meeting of the North American Chapter of the International Group for the Psychology of*

Mathematics Education (3-26). Columbus, OH: Clearinghouse for Science, Mathematics, and Environmental Education.

- Duval, R. (1988). Graphiques et equations: l'articulation de deux registres. *Annales de Didactique et de Sciences Cognitives*, 1, 235-255.
- Eves, H. W. (1997). *Foundations and fundamental concepts of mathematics*. Mineola, New York: Dover Publications, Inc.
- Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, Netherlands: Kluwer Academic Publishers.
- Fischbein, E., Jehiam, R. & Cohen, C. (1995). The concept of irrational number in high-school students and prospective teachers. *Educational Studies in Mathematics*, 29, 29-44. DOI: 10.1007/bf01273899
- Giannakoulias, E., Souyoul, A. & Zachariades, T. (2007). Students' thinking about fundamental real numbers properties. In: Pitta-Pantazi, D. & Philippou, G. (Eds.). *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (426–425). Cyprus: ERME, Department of Education, University of Cyprus.
- Glaeser, G. (1981). Epistémologie des nombres relatifs. *Rech Didact Math*, 2-3, 303-346.
- Gray, E. & Doritou, M. (2008). The number line: Ambiguity and interpretation. In: Figueras, O., Cortina, J. L., Alatorre, S., Rojano, T. & Sepu´lveda, A. (Eds.). *Proceedings of the joint Meeting of PME 32 and PME-NA XXX Vol. 3* (97-104). Mexico: Cinvestav-UMSNH.
- Hannula, M. S. (2003). Locating fractions on a number line. In: Pateman, N. A., Dougherty, B. J. & Zilliox, J. T. (Eds.). *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA Vol. 3* (3-24). Honolulu, HI: PME.
- Hannula, M. S., Pehkonen, E., Maijala, H. & Soro, R. (2006). Levels of students' understanding on infinity. *Teaching Mathematics and Computer Science*, 4 (2), 317-337. DOI: 10.5485/TMCS.2006.0129
- Heefer, A. (2011). Historical objections against the number line. *Science & Education*, 20 (9), 863-880. DOI: 10.1007/s11191-011-9349-0
- Katz, U. K. & Katz, G. M. (2011). Stevin Numbers and Reality. *Foundations of Science*, 17 (2), 109-123. DOI: 10.1007/s10699-011-9228-9
- Kidron, I. (2016). Understanding irrational numbers by means of their representation as non-repeating decimals. In: Nardi, E., Winsløw, C. & Hausberger, T. (Eds.). *First conference of International Network for Didactic Research in University Mathematics* (73-83). Mar 2016, Montpellier, France. hal-01337883f.
- Lemonidis, C. (1990). *Conception, réalisation et résultats d'une expérience d'enseignement de l'homothétie* (Thèse de Doctorat). Université Louis Pasteur, I.R.E.M. de Strasbourg. DOI: 10.12681/eadd/3786
- Lemonidis, C. (1991). Analyse et réalisation d'une expérience d'enseignement de l'homothétie. *Recherches en Didactique des Mathématiques (R.D.M)*, 11 (2-3), 295-324.

- Lemonidis, C., Tsakiridou, H. & Meliopoulou, I. (2018). In-Service Teachers' Content and Pedagogical Content Knowledge in Mental Calculations with Rational Numbers. *International Journal of Science and Mathematics Education*, 16 (6), 1127–1145. DOI: 10.1007/s10763- 017-9822-6.1
- Lowrie, R., & Diezmann, C. M. (2005). Fourth-grade students' performance on graphical languages in mathematics. In: Chick, H. L. &. Vincent, J. L (Eds.). *Proceedings of the 30th Annual Conference of the International Group for the Psychology of Mathematics Education Vol 3* (265–272). Melbourne: PME.
- Mitchell, A. & Horne, M. (2008). Fraction number line tasks and the additivity concept of length measurement. In: *Proceedings of the 31st annual conference of the mathematics education research group of Australasia* (353–360).
- Mpantes, G. (2013). *[Dedekind, continuity and infinity, cuts Dedekind] O Dedekind, η συνέχεια και το άπειρο, τομές Dedekind*. Retrieved from: https://www.scribd.com/doc/147611828/ last access to 21/06/2019. [in Greek]
- Murphy, C. (2011). Comparing the use of the empty number line in England and the Netherlands. *British Educational Research Journal*, 37 (1), 147–161. DOI: 10.1080/01411920903447423
- Murphy, C. (2008). The use of the empty number line in England and the Netherlands. *Proceedings of PME*, 32 (4), 9–16.
- Neal, K. (2002). *From discrete to continuous. The broadening of number concepts in early modern England*. Melbourne: R.W Home, University of Melbourne.
- Núñez, R. E. (2011). No innate number line in the human brain. *Journal of Cross-Cultural Psychology*, 42 (4), 651–668. DOI: 10.1177/0022022111406097
- Pearn, C. & Stephens, M. (2007). Whole number knowledge and number lines help develop fraction concepts. In: Watson, J. & Beswick, K. (Eds.). *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia (MERGA): Mathematics: Essential research, essential practice Vol. 2* (601–610). Hobart, Sydney: MERGA.
- Petit, M. M., Laird, R. E., Marsden, E. L. & Ebby, C. B. (2010). *A focus on fractions: Bringing research to the classroom*. London: Routledge.
- Petitto, A. (1990). Development of number line and measurement concepts. *Cognition and Instruction*, 7 (1), 55–78. DOI: 10.1207/s1532690xci0701_3
- Pycior, H. (1987). British abstract algebra: development and early reception. *Cahiers d'Histoire et de Philosophie des Sciences*, 21, 152–168.
- Roque, T. (2012). *Historia da matemαtica: uma visyo critica, desfazendo mitos e lendas*. Rio de Janeiro: Jorge Zahar.
- Saxe, G. B., Shaughnessy, M. M., Shannon, A., Langer-Osuna, J. M., Chinn, R. & Gearhart, M. (2007). Learning about fractions as points on a number Line. In: Martin, W. G., Strutchens, M. E. & Elliott, P. C. (Eds.). *The learning of mathematics: Sixty-ninth yearbook* (221–237). Reston, VA: National Council of Teachers of Mathematics.

- Schubring, G. (2005). *Conflicts between generalization, rigor, and intuition. Number concepts underlying the development of analysis in 17–19th century France and Germany.* New York: Springer. DOI: 10.1007/0-387-28273-4
- Schubring, G. (1986). Ruptures dans le statut mathématique des nombres négatifs. *Petit x*, 12, 5–32.
- Sinkevich, G. (2015). *On the History of Number Line*. Saint Petersburg: Department of Mathematics, Saint Petersburg State University of Architecture and Civil Engineering. Retrieved January 18, 2020 from www: https://www.academia.edu/20922833/On_the_history_of_number_line.
- Sirotic, N. & Zazkis, R. (2007). Irrational numbers on the number line – where are they? *International Journal of Mathematical Education in Science and Technology*, 38 (4), 477–488. DOI: 10.1080/00207390601151828
- Skoumpourdi, C. (2010). The number line: An auxiliary means or an obstacle? *International Journal for Mathematics Teaching and Learning* (Electronic Journal).
- Struik, D. J. (1982). [A Brief History of Mathematics] *Συνοπτική ιστορία των Μαθηματικών*. Αθήνα: Ζαχαρόπουλος. [in Greek]
- Teppo, A. & Van den Heuvel-Panhuizen, M. (2014). Visual representations as objects of analysis: the number line as an example. *ZDM*, 46 (1), 45–58. DOI: 10.1007/s11858-013-0518-2
- Thomaidis, Y. (1995). *Didactic transposition of mathematical concepts and learning obstacles (The case of absolute value)* (doctoral thesis). Greece: Aristotle University of Thessaloniki. [in Greek]
- Thomaidis, Y. & Tzanakis, C. (2007). Historical evolution and students' conception of the order relation on the number line: the notion of historical "parallelism" revisited. *Educational Studies in Mathematics*, 66, 165–183. DOI: 10.1007/s10649-006-9077-6
- Tirosh, D. & Stavy, R. (1996). Intuitive rules in science and mathematics: The case of "everything can be divided by two". *International Journal of Science Education*, 18, 669–683. DOI: 10.1080/0950069960180603
- Treffers, A. (1993). Wiskobas and Freudenthal: realistic mathematics education. *Educational Studies in Mathematics*, 25 (1–2), 89–108. DOI: 10.1007/978-94-017-3377-9_6
- Van den Heuvel-Panhuizen, M. (2008). Learning from "Didactikids": An impetus for revisiting the empty number line. *Mathematics Education Research Journal*, 20 (3), 6–31. DOI: 10.1007/bf03217528
- Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, 54 (1), 9–35. DOI: 10.1023/b:educ.0000005212.03219.dc
- Van der Waerden, B. L. (1985). *A history of algebra. From al-Khwarizmi to Emmy Noether*. Berlin: Springer-Verlag.
- Vamvakoussi, X. & Vosniadou, S. (2012). Bridging the gap between the dense and the discrete: The number line and the "rubber line" bridging analogy. *Mathematical Thinking and Learning*, 14 (4), 265–284. DOI: 10.1080/10986065.2012.717378

- Vamakoussi, X. & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, 28, 181-209. DOI: 10.1080/07370001003676603
- Vamvakoussi, X. & Vosniadou, S. (2007). How many numbers are there in an interval? Presuppositions, synthetic models and the effect of the number line. In: Vosniadou, S., Baltas, A. & Vamvakoussi, X. (Eds.). *Reframing the conceptual change approach in learning and instruction* (267-283). Oxford, UK: Elsevier.
- Vamvakoussi, X. & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, 14, 453-467. DOI: 10.1016/j.learninstruc.2004.06.013
- Waerden, B. L. (1985). *A history of algebra: from Al-Khwārizmī to Emmy Noether*. Springer.
- Wilder, L. R. (1986). *Εξέλιξη των μαθηματικών εννοιών.* The Open University. Αθήνα: Π. Κουτσουμπός Α. Ε. [The evolution of mathematical concepts] [in Greek]