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Modeling the Equivalence of Mathematical Expressions in Initial Teaching[2]

Extended summary

One of the dominant problems that students have when switching from arithmetic to algebra is an insufficient and limited understanding of the equivalence of expressions (Chaiklin & Lesgold, 1984; Linchevski & Livneh, 1999; Kieran et al., 2013). Equivalence of expressions is based on the knowledge and application of arithmetic rules and properties of operations. However, numerical equations expressing the properties of operations and general rules of arithmetic are often understood by students solely as a command to calculate the value of an expression (Booth, 1988; Sfard, 1991; Linchevski & Livneh, 1999; Malara & Iaderosa, 1999; Kieran, 2004; Linchevski & Livneh , 2007). In this way, most students develop only a procedural understanding that inhibits their algebraic thinking abilities (Sfard & Linchevski, 1994; Crowley et al. 1994; Kieran 1996; Kieran, 2004). Equivalence can be introduced using numerical expressions, but without calculating the value of the expression, or using algebraic (letter) expressions. Malara and Iaderosa (1999) pointed out a problem that is reflected in the fact that when students use letters instead of numbers, they often do not recognize the properties they knew in arithmetic. These authors believe that the early introduction of the variable and algebraic aspects into arithmetic could later improve the understanding of algebraic notation. The support for

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the semantics of syntactic manipulations can be found in contextual problems (i.e., the modeling process) and/or representations (Cai, 2014; Chazan, 2000; Stylianou, 2011).

The aim of the research is to determine the effects of different approaches to the introduction of equivalence of expressions. Two modules of systematization of algebraic laws and properties of operations using expression equivalence have been designed. The first included algebraic expressions as abstract representations, which were concretized by schematic representations, while the second model used numerical expressions as concrete representations that were abstracted by rhetorical generalizations. In both modules, mathematical modeling was used to support the understanding of equivalence. In this paper, there are two research questions: 1. Does mathematical modeling affect student achievement in transforming expressions into equivalent forms? Within this question, we also consider the influence of the use of visual representations and rhetorical generalizations on the understanding of the process of expression transformation; 2. Does the abstractness of language and the use of algebraic symbolism affect the understanding of equivalence expressed through student achievement? Participants in this research were fourth-grade students (10-11 years). The total number of the participants was 148 (6 classes). Based on the data collected in the initial test, we formed three groups of students: two that were included in the experiment (E1 and E2) and the control group (K).

The results show that there is a statistically significant difference in terms of achievement between the experimental and control groups, which implies that the modeling process in which textual tasks are used as a starting point and a framework for the meaning of transformation is an efficient methodological procedure for developing meaning and applying arithmetic rules. There is no statistically significant difference between the students who were taught using letter or numerical expressions, which is in contrast to the research results considered in the theoretical basis: 1) students will be more successful when working with numerical expressions; 2) algebraic language as a way of expressing generalization is an obstacle in learning and 3) structural understanding of expression and equality is a problem for much older students as well. We believe that students who worked on the transformation of the letter expressions during the experiment bridged the abstractness of language by using schemes as bearers of meaning, and students who worked on the transformation of the numerical expressions generalized their actions with rhetorical generalizations.

Although it would be expected that the students who were taught to use letter expressions would be more successful on the tasks that contain them, in our research this is not the case. On the group of tasks that contain letter expressions, the success in writing two or more expressions on some tasks is identical, and on some the success of the group taught with numerical expressions is higher, which is contrary to the previous research (Cerulli & Mariotti, 2001; Malara & Iaderosa, 1999; Stacey & MacGregor, 1999). This result shows that the success of students in the transformation of letter expressions depends on an essential understanding of the meaning of the rules, and not exclusively on the means by which generalizations are expressed.

The number of structural errors is significantly higher on the tasks with numerical expressions in a symbolic context compared to textual tasks. This shows that even in a situation where students can check the accuracy of equality by calculation, the number of errors increases when there is no context and meaning on which the transformations are based (Banerjee, Subramaniam & Naik, 2008; Booth, 1988; Linchevski & Livneh, 1999; Subramaniam & Banerjee 2004). This result has significant implications. The determining factor of success in the transformations of equivalent expressions is not the question of algebraic and arithmetic language, but the development of the meaning of relations through the process of modeling. A well-chosen realistic context gives students the opportunity to explore and expand their knowledge about the properties of a set of natural numbers.

Keywords: equivalence of mathematical expressions, modeling, mathematical symbolism, algebra.

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